Depth-Robust Graphs and Their Cumulative Memory Complexity

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Moderately Hard Function

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1. Computable by honest party.
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- Password Based Cryptography
  - Password Hashing (E.g. Login Server)
  - Key Derivation Functions
- Proofs-of-Effort
  - Distributed PoW for Consensus (E.g. Ethereum, Lightcoin, Dogecoin, etc.)
  - Against SPAM [ABMW05, DGN03, DNW05]
  - Against Sybil attacks.

Why “Memory” Hard?

In practice cost-effective brute-forcing often uses GPUs, FGPAs & ASICs.

- Bitcoin miners, DES Cracker [EFF98], Sagitta Password Cracker, etc.

- Why? ASICs provide a financial incentive.
  - Specifically: Computation is cheaper for custom hardware (e.g. ASICs) then general purpose CPUs.

- Want: More egalitarian notion of “hardness” than computation.

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  1. Can be computed in sequential time n.
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iMHF advantage: Implementations easier to secure against certain cache-timing attacks.

- Important for some password based crypto applications.

iMHFs

- Password Hashing Competition
  - Winner: Argon2i [BDK15]
  - Finalists: Catena[FLW15], Lyr2 [SAASB15], Pomelo [W15],…
  - Other contestants: Rig-v2 [CJMS14], Gambit [P14], TwoCats [C14],…
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- Usually designed based on intuition and verified via cryptanalysis.
- Exceptions: Catena, Balloon Hashing, AS15
  - Balloon Hashing has security proof for sequential adversaries in ROM.
  - AS15 has proof for parallel adversaries in ROM.

Amortization and Parallelism

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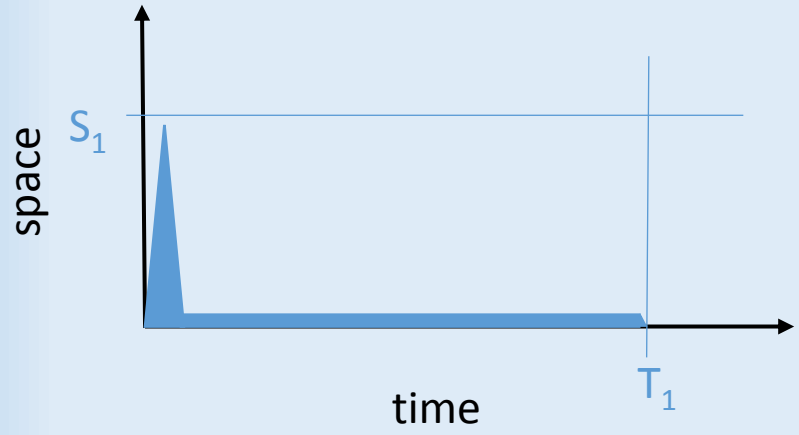

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cost of computing
f once

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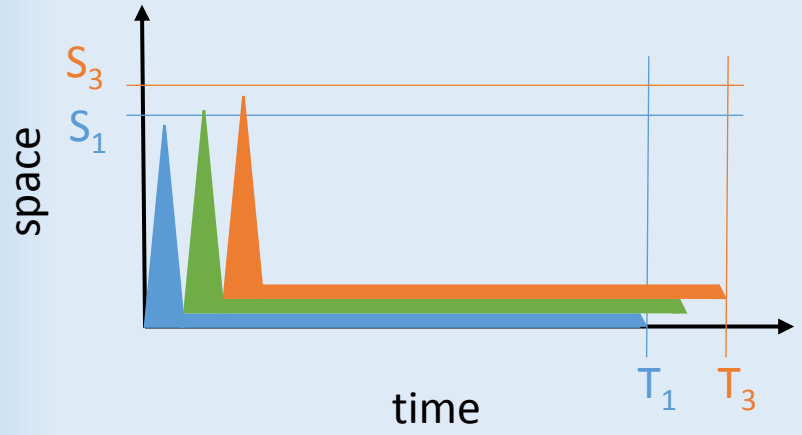

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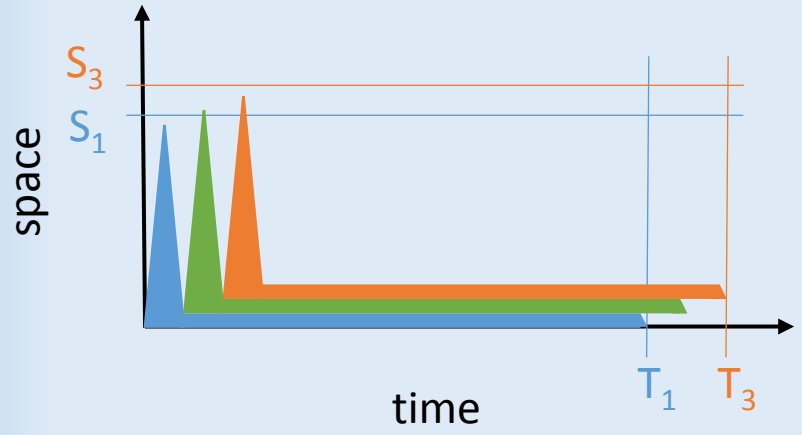

$$ST_1 = S_1 \times T_1 \approx S_3 \times T_3 = ST_3$$

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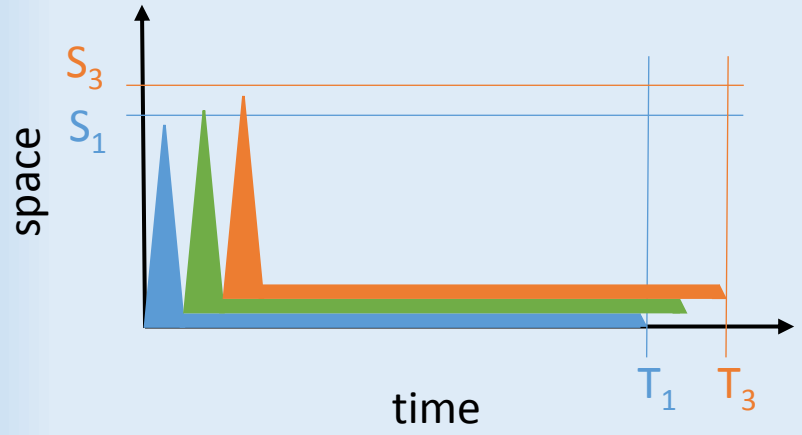
Amortization and Parallelism

Problem: function f_n (consisting of n RO calls) such that: $ST\ f \times n\ = O(ST\ f)$

$n\,n\ \times n\ n\ n \times n \times n\ f \times n$



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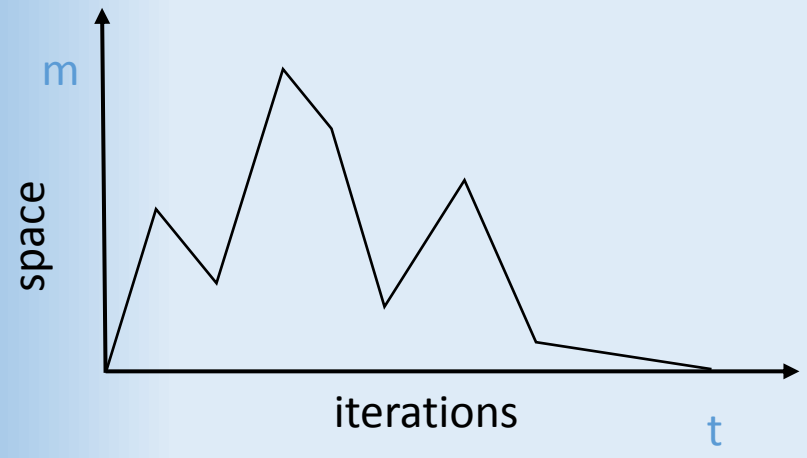
[AS15] ∃ function f_n (consisting of n RO calls) such that: $ST\left(f^{\times\sqrt{n}}\right) = O(ST(f))$

Cumulative Memory Complexity

- Fix an execution...



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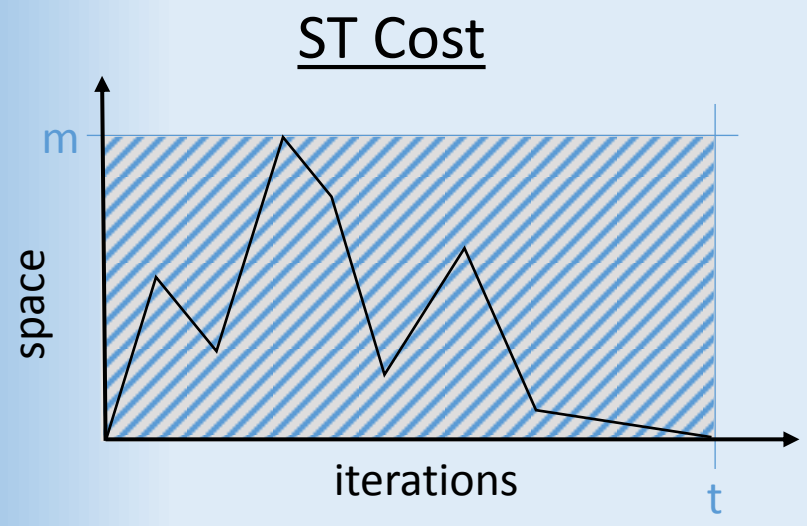
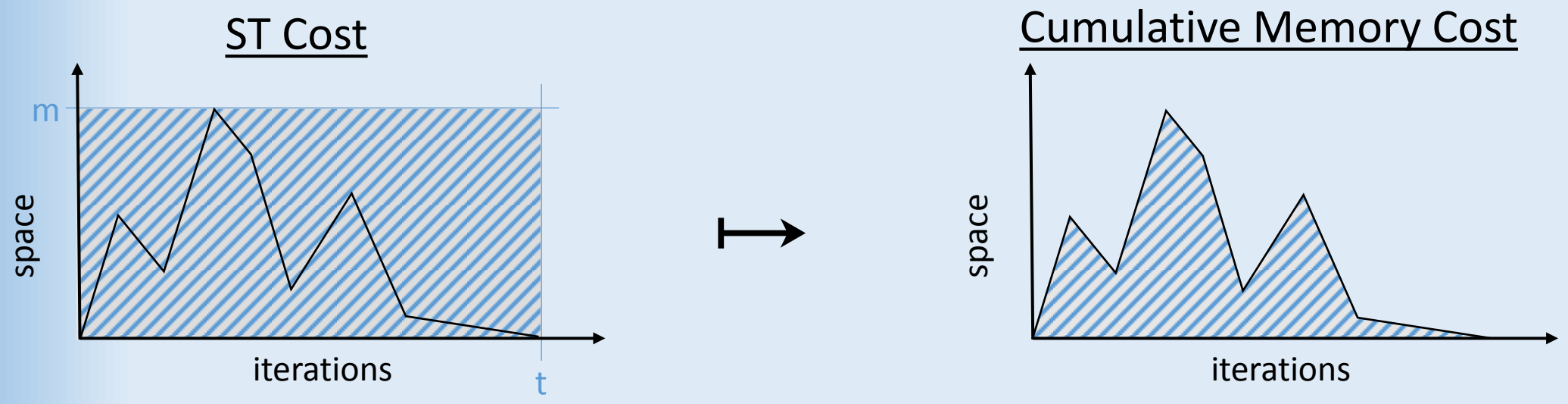


Cumulative Memory Complexity

- Fix an execution...
- Idea: Define the cost to be area under the “memory curve”.



Parallel Pebbling Game

- Intuition: Models Parallel Computation

- Iteratively place pebbles on the nodes of DAG *G*.
  - Initially no pebbles on *G*. Each node can have at most one pebble.

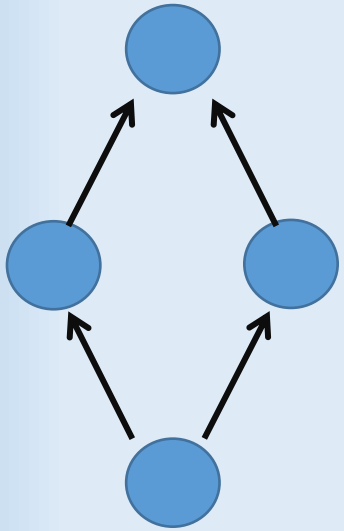
- Goal: Place a pebble on sink node(s) of *G*.

- Rules:
  1. Can place a pebble on *v* only if all of parents of *v* currently have a pebble.
     $\Rightarrow$ can always place a pebble on source nodes
  2. Can remove any pebble at any time.

A New Parallel Pebbling Game

Parallel Pebbling Game: Same as Black Pebbling, except can touch many pebbles per iteration.

Complexity: Cumulative Pebbling Complexity (CPC).



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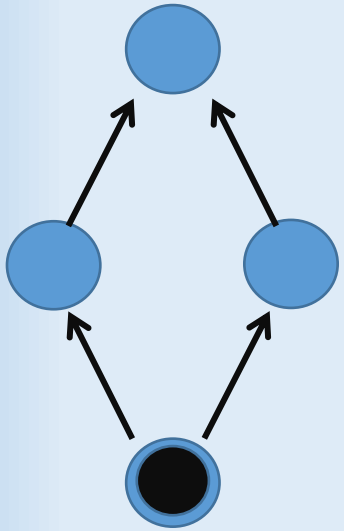

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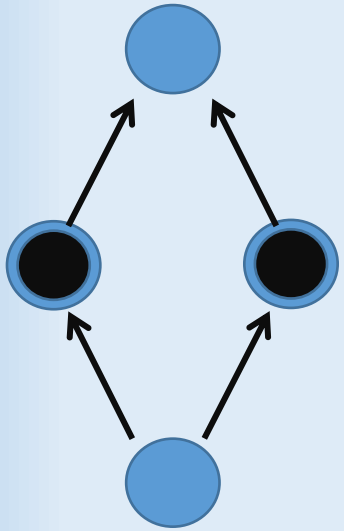

CPC-cost = 1+ 2+

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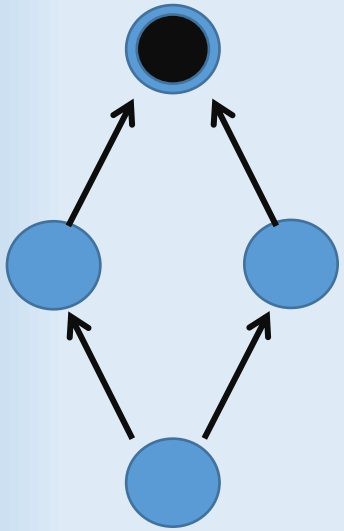

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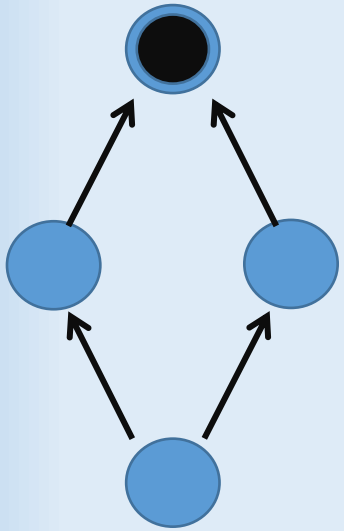

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CPC(Graph G) := min CPC(Pebbling of G)

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- Theorem [AS15]
  - Let $\mathcal{H}: \{0,1\}^{2w} \rightarrow \{0,1\}^{w}$ be a RO and G be a DAG.
  - Let f be the function given by (G, $\mathcal{H}$).

$$\Longrightarrow CMC(f) \geq CPC(G)/4$$

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- New Goals:
  - Security Proofs: find <u>constant indegree</u> graph with high CPC.
  - Attacks: find low CPC pebbling strategies.

Some Previous Results About CPC

MHF	Upper Bound	Lower Bound
Argon2i-A Balloon Hashing	$\tilde{O}(n^{1.75})$ [AB16]	–
Argon2i-B	$O(n^{1.8})$ [AB17]	–
Catena	$O(n^{1.67})$ [AB16]	–
AS15	–	$\Omega(n^2/\log^{10}(n))$ [AS15]
Any iMHF	$O\left(\frac{n^2 \mathrm{loglog}(n)}{\log(n)}\right)$[AB16]	–
SCRYPT (dMHF)	$O(n^2)$	–

New Results

MHF	Upper Bound	Lower Bound
Argon2i-A Balloon Hashing	$\tilde{O}(n^{1.71})$ [This Work] ~~$\tilde{O}(n^{1.75})$~~ [AB16]	$\tilde{\Omega}(n^{1.\bar{6}})$ [This Work]
Argon2i-B	$O(n^{1.8})$ [AB17]	$\tilde{\Omega}(n^{1.\bar{6}})$ [This Work]
Catena	$\tilde{O}(n^{1.618})$ [This Work] ~~$O(n^{1.67})$~~ [AB16]	$\tilde{\Omega}(n^{1.5})$ [This Work]
AS15	–	$\Omega(n^2/\log^{10}(n))$ [AS15]
This Work	–	$\Omega(n^2/\log(n))$
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SCRYPT (dMHF)	$O(n^2)$	$\Omega(n^2)$ [Next Talk]

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Depth-Robust Graphs

A directed ac(n), Ω(n))-depth-robust with indegree O(log(n)).

- G called “(e,d) -reducible” iff G is <u>not</u> (e,d)-depth-robust.

History:

- First considered: Erdös, Graham, Szemerédi [EGS77]
- EGS graph: (Ω(n), Ω(n))-depth-robust with indegree O(log(n)).

New Construction: Technique

Lemma "Indegree Reduction": If G has indegree δ and is (e,d)-depth-robust then there exists H with indegree 2 and:

size(H) $\leq 2\delta$*size(G) H is (e,dδ)-depth-robust

heorem: Let G=(V,E) be (e,d)-depth-robust then $\mathrm{CPC}(G) > ed$.

Corollary: There is a DAG G with maximum indegree $\delta = 2$ and $\mathrm{E_R}(G) = \Omega\left(\frac{n^2}{\log n}\right)$. Furthermore, there is a sequential pebbling algorithm N with cost $\mathrm{E_R}(N) = O\left(\frac{n^2}{\log n}\right)$.

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New Bounds For Known Constructions

- New Lower Bounds
  - Lower bound Depth-Robustness of Balloon Hashing and Argon2i-{A,B}
  - Second Technique: “Dispersal” Property of a DAG
    1. Lower Bound CPC(G) in terms of Dispersal properties of G
    2. Analyze Dispersal properties of Catena (Dragonfly and Butterfly versions)

- New Upper Bounds
  - Idea: Recursive version of AB16 Algorithm
  - Analyze Depth-Robustness “curves” for Argon2i, Catena & Balloon Hashing

The CPC of Depth-Robust Graphs

- **Theorem**: Let G=(V,E) be (e,d)-depth-robust then $\mathrm{CPC}(G) > ed$.

"The proof of this result is really an incredibly simple and beautiful two-line argument….I generally view simplicity as a positive, and this is the right proof. But there is such as thing as too simple..."

Proof by Picture

Given: DAG G = (V,E)
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Fix an optimal pebbling of G: $P = (P_1, P_2, P_3, \dots)$

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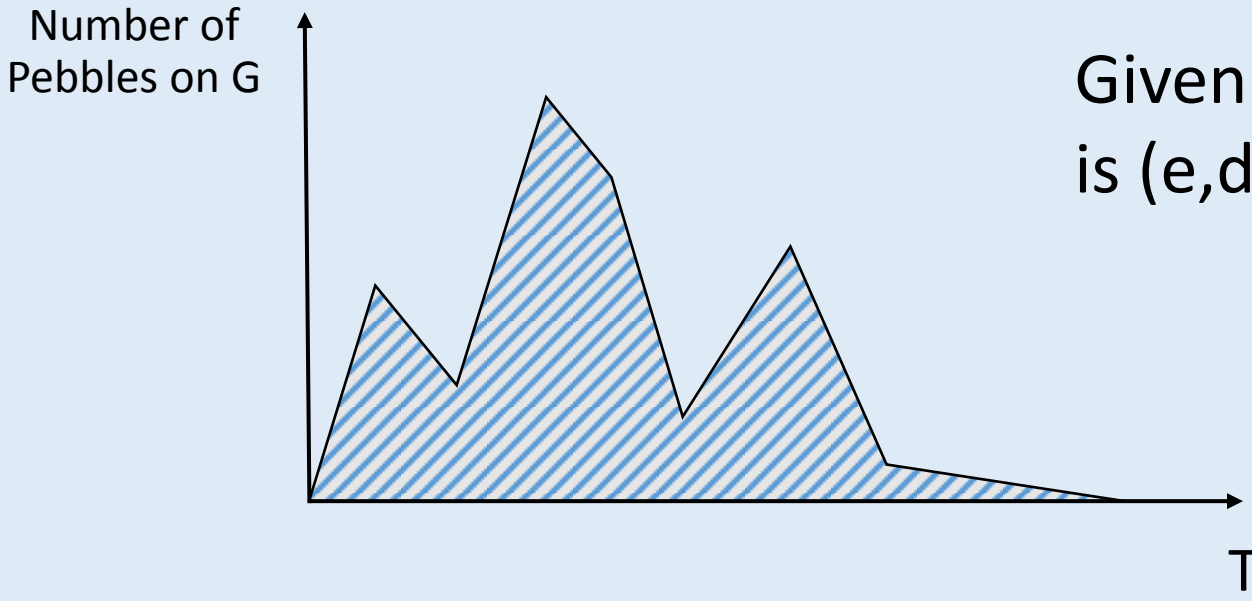

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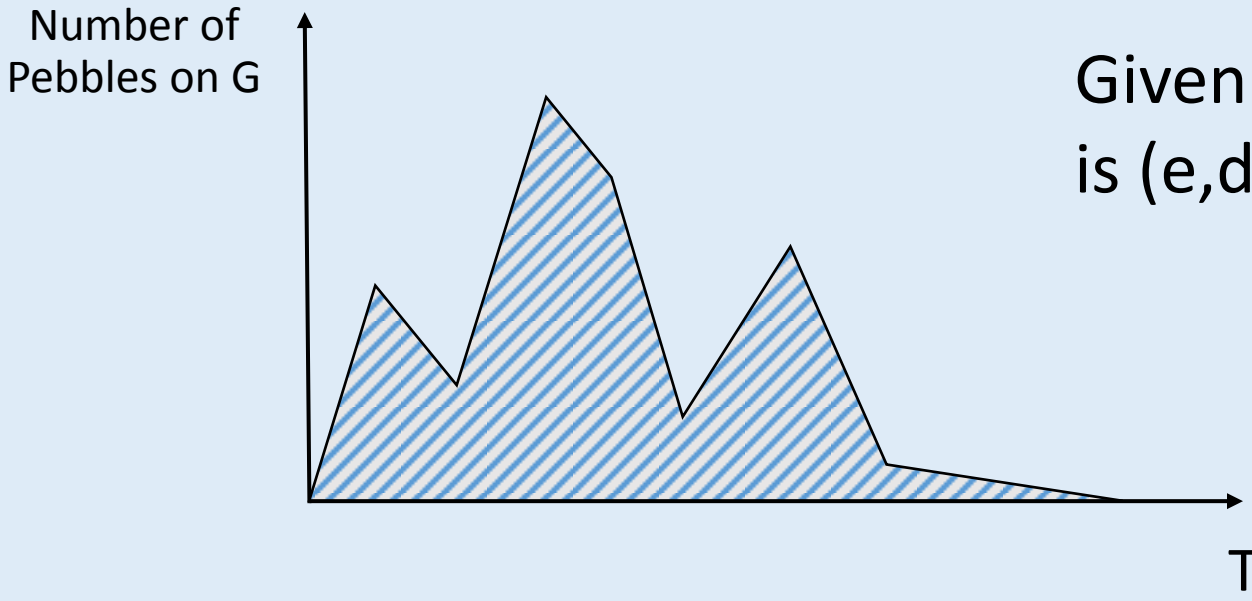

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Fact 1: Area under curve
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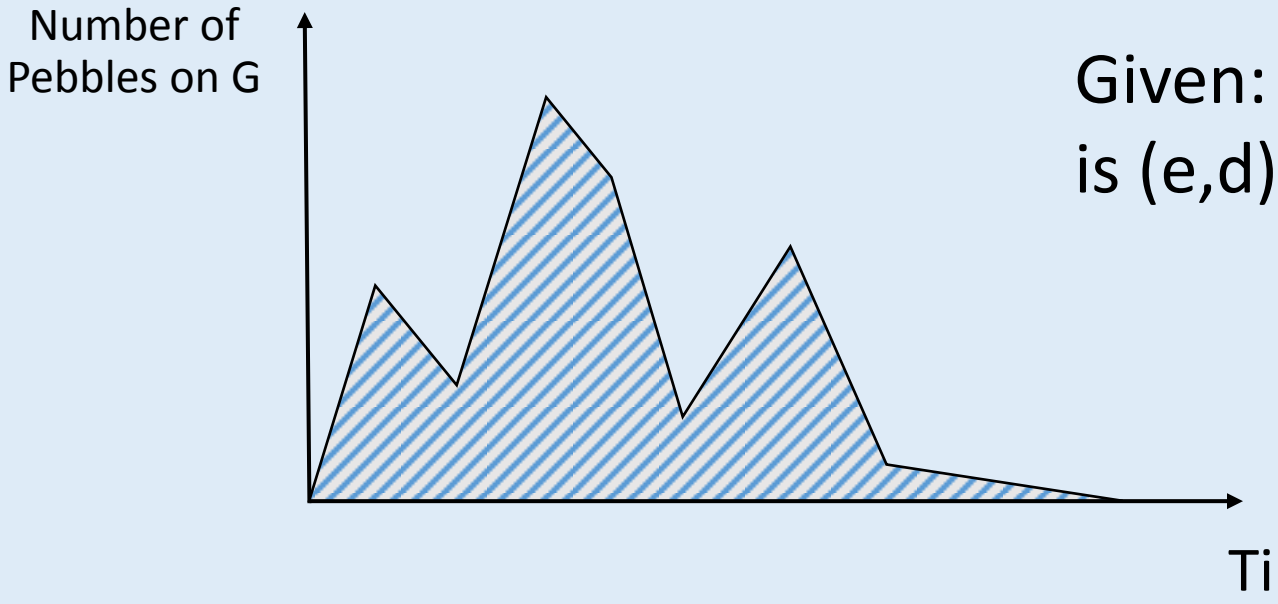

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Fact 1: Area under curve = CPC (G)

Fact 2: P pebbles every node in G at least once.

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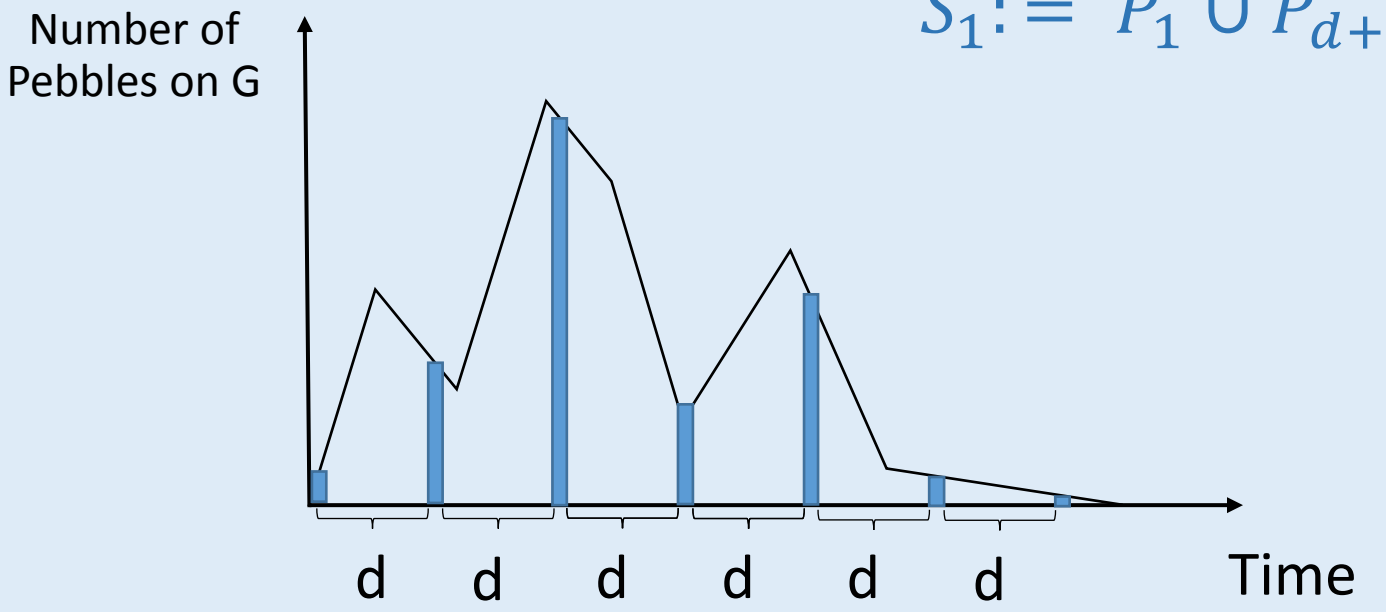
Proof by Picture

$S_1 := P_1 \cup P_{d+1} \cup P_{2d+1} \cup \ldots \subseteq V$



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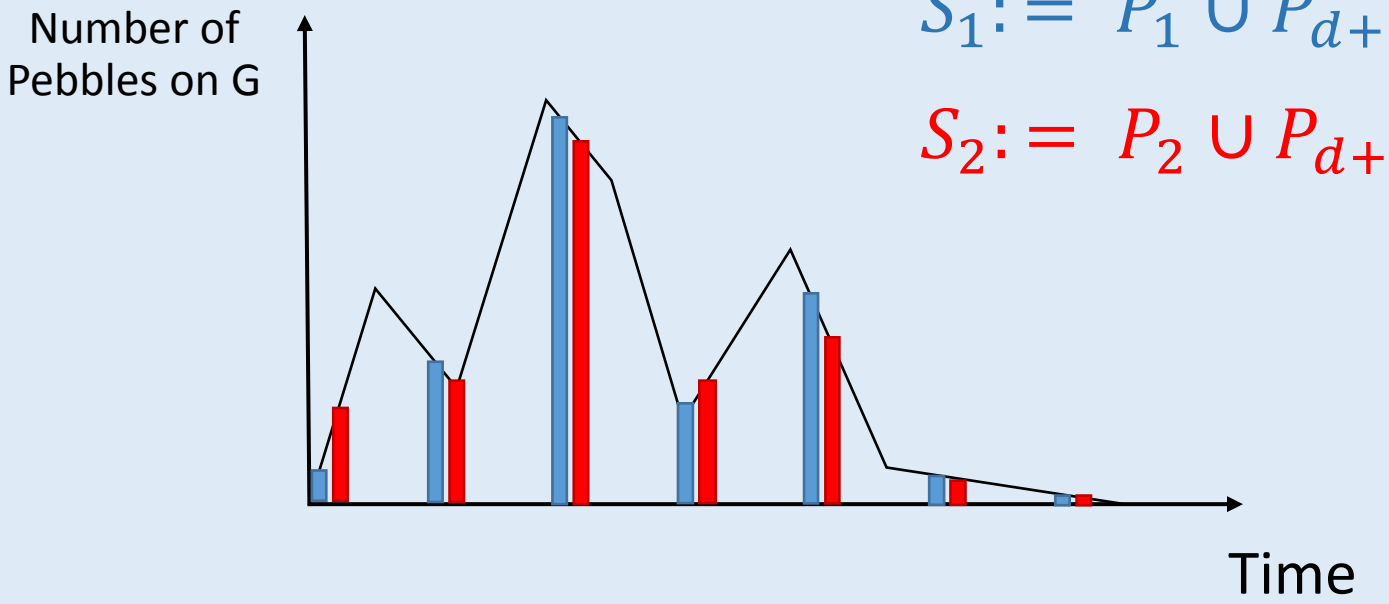

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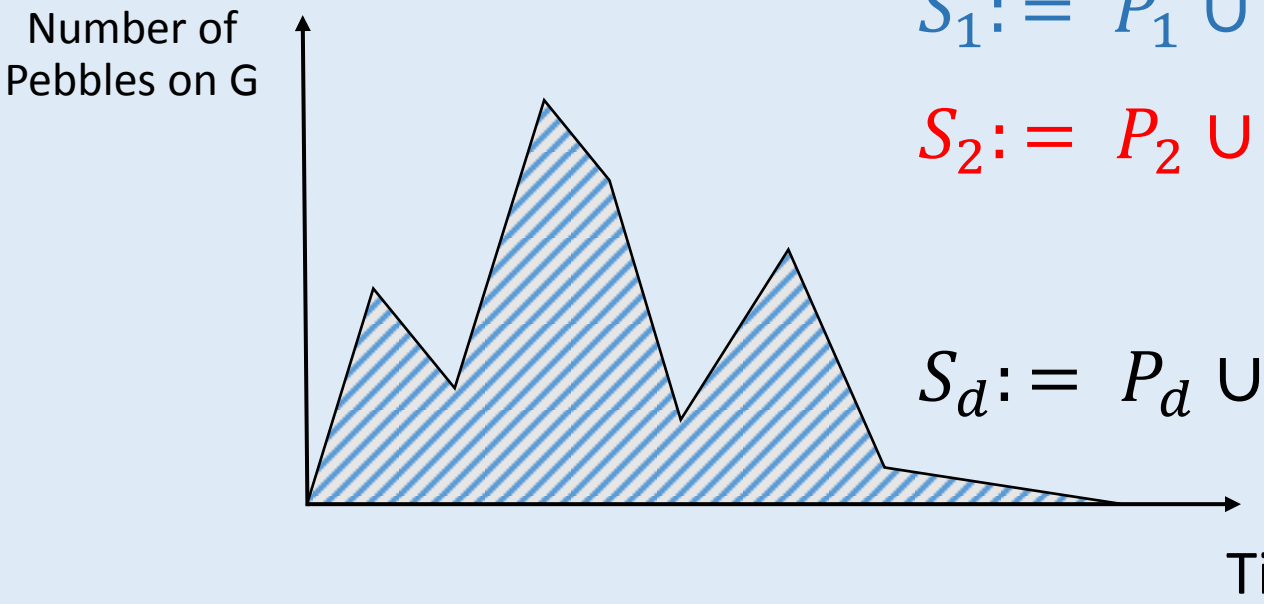

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$S_d := P_d \cup P_{2d} \cup P_{3d} \cup \ldots \subseteq V$

$$\sum_{i=1}^{i=d} |S_i| \leq CPC(G)$$

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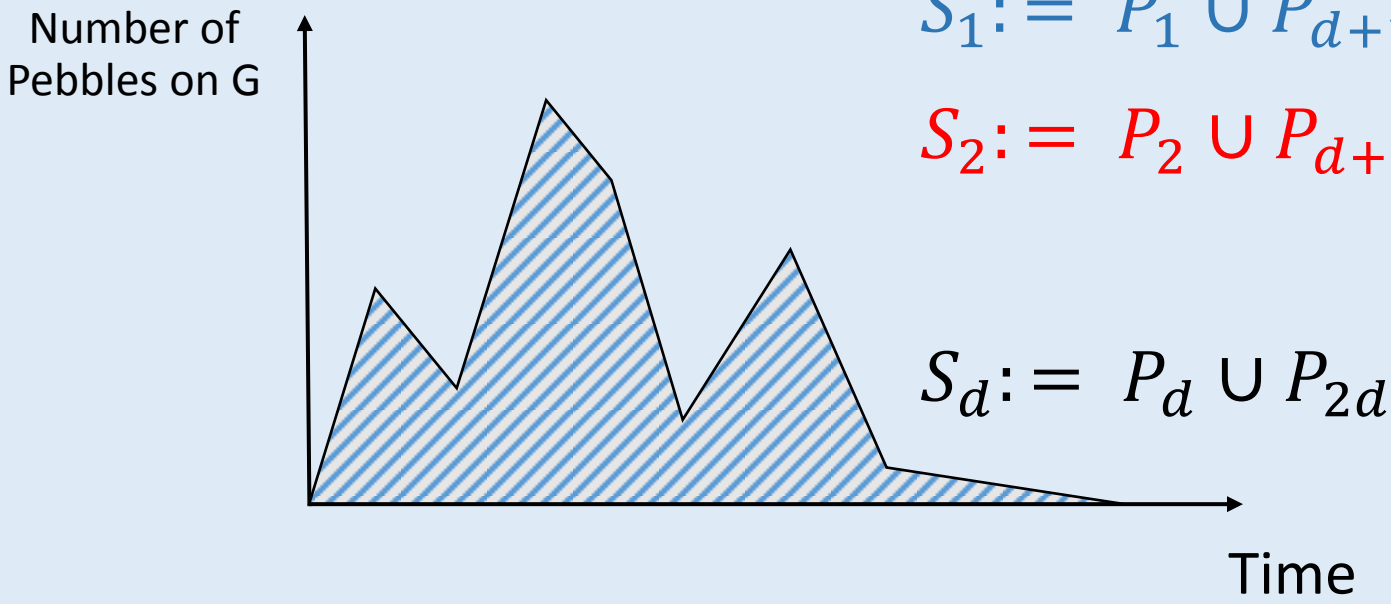

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$$S_d := P_d \cup P_{2d} \cup P_{3d} \cup \ldots \subseteq V$$

$$\Rightarrow \exists i \text{ such that } |S_i| \leq \frac{CPC(G)}{d}$$

Proof by Picture

Let graph G’ = G with nodes in S_i removed.

- Let pebbling P' = P but with $P_{i+jd} = \emptyset \ \ \forall j$.

P’ legally pebbles every node in G’ at least once.

Notice: Every d steps P’ has no pebbles on G’.

$\Rightarrow$ No path in G’ is longer than d-1.

$\Rightarrow \ e < \ |S_i| \leq \frac{CPC(G)}{d}$

$\Rightarrow \ ed < CPC(G)$.

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⇒ $ed < CPC(G)$.

Fact 2: P pebbles every node in G at least once.

Proof by Picture

P’ legally pebbles every node in G’ at least once.

Let graph G’ = G with nodes in S_i removed.

- Let pebbling P' = P but with $P_{i+jd} = \emptyset \ \ \forall j$.

$\Rightarrow$ P’ legally pebbles every node in G’ at least once.

Notice: Every d steps P’ has no pebbles on G’.

$\Rightarrow$ No path in G’ is longer than d-1.

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Fact 3: Pebbling a path of length d takes at least d time.

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But G is (e,d)-depth robust

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Thank You